ON ORLICZ SEQUENCE SPACES. II

BY

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ABSTRACT

Given a separable Orlicz sequence space l_F we investigate those Orlicz sequence spaces l_f which are isomorphic to subspaces (respectively complemented subspaces) of l_F. We give in particular an example of a reflexive Orlicz sequence space which does not contain any l_p, $1 < p < \infty$, as a complemented subspace.

1. Introduction

The present paper is a continuation of [4]. Its main purpose is to investigate the structure of those subspaces of an Orlicz sequence space which are themselves Orlicz sequence spaces. We assume that the reader is familiar with [4] and in particular with the basic notions related to Orlicz sequence spaces which were reviewed in Section 2 of [4]. Unless stated otherwise we assume that the Orlicz functions considered here satisfy the Δ_2 condition (at 0).

The first result to be proved here (Theorem 1, Section 2), gives a necessary and sufficient condition on f for l_f to be isomorphic to a subspace of a given Orlicz space l_F. The condition is that f should be equivalent to a function of the compact convex set $C_{F,1}$ in $C(0,1)$ (recall that $C_{F,t} = \overline{\operatorname{conv}}_{0<s\leq t}\{F(sx)/F(s)\}$).

While the solution to this problem is relatively simple, it seems to be more difficult to find a characterization of those f such that l_f is isomorphic to a complemented subspace of l_F. From the existence of the so called "averaging projections" (or conditional expectations) it follows immediately that if f is equivalent to a function of $E_{F,1} = \overline{\{F(sx)/F(s)\}}_{0<s\leq 1}$, then l_f is isomorphic to a complemented subspace of l_F. Is the converse true? We do not know the answer. The main result of Section 2 is a partial answer to this question. We show that if f is "strongly non-equivalent" to $E_{F,1}$ then l_f is indeed not isomorphic to a complemented

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subspace of l_F. The requirement that f be strongly non-equivalent to $E_{F,1}$ is a quantitative uniformity condition which is stronger then the requirement that f is not equivalent to any function of $E_{F,1}$.

Section 3 is devoted mainly to the study of some examples. The first two examples we consider are Orlicz spaces l_F in which F is given by an explicit formula.These examples, besides illustrating results of Section 2 and [4], show also the role of duality arguments in the questions we consider in this paper. The next example considered in Section 3 is the Orlicz space constructed in [4], in the proof of Theorem 3 there. We show, by using the main result of Section 2, that this space does not have any l_p-space as a complemented subspace. To the best of our knowledge this is the first known example of a Banach space which contains no complemented subspace isomorphic to c_0 or to some l_p, $1 \leqq p \leqq \infty$. Though this space l_M is constructed in a somewhat "artificial" manner it is nevertheless a "nice" Banach space from many points of view. It has a symmetric basis and is reflexive (actually even uniformly convexifiable).

The rest of Section 3 is devoted mainly to the investigation of some classes of Orlicz sequence spaces. These are the "minimal" Orlicz spaces, some spaces obtained by a composition operation from other Orlicz sequence spaces and universal Orlicz sequence spaces. All these examples and classes illustrate the very rich structure of Orlicz sequence spaces. We are convinced that Orlicz sequence spaces will play in the future an important role as test spaces for various questions in the isomorphic theory of Banach spaces. The structure of Orlicz function spaces is naturally more involved (and interesting) than that of sequence spaces. We plan to treat the structure of subspaces of Orlicz function spaces and the connection between Orlicz function and sequence spaces in a future paper.

The last section of this paper is devoted to a discussion of some questions, concerning Orlicz sequence spaces and more general spaces with symmetric bases, which arise naturally from the results of Sections 2 and 3.

2. General results

We begin by proving a theorem which characterizes the functions f such that l_f is isomorphic to a subspace of a given Orlicz sequence space l_F.

THEOREM 1. *Let F be an Orlicz function satisfying the Δ_2 condition. An Orlicz sequence space l_f is isomorphic to a subspace of l_F if and only if f is equivalent to some function in* $C_{F,1} = \overline{\operatorname{conv}}_{0<s\leqq 1}\{F(sx)/F(s)\}$.

PROOF. The necessity has been proved by Lindberg [3]. Let us briefly recall the proof. Let T be an isomorphism from l_f into l_F and let $\{u_i\}$ and $\{e_i\}$ denote the unit vector bases of these spaces. By a standard procedure one can choose a subsequence $\{u_{i_n}\}$ of $\{u_i\}$ and a sequence of unit vectors $\{v_n\}$ in l_F such that $\| v_n - Tu_{i_n}/\| Tu_{i_n} \| \| \leqq 2^{-n}$, and $v_n = \sum_{i=p_n+1}^{p_{n+1}} \lambda_i e_i$ for suitable $\{\lambda_i\}$ and a suitable increasing sequence $\{p_n\}$ of integers.* Let $g_n(x) = \sum_{i=p_n+1}^{p_{n+1}} F(\lambda_i x)$, $n = 1,2,\cdots, 0 \leqq x \leqq 1$. Since $1 = \| v_n \| = \sum_{i=p_n+1}^{p_{n+1}} F(\lambda_i)$, it follows that $g_n \in C_{F,1}$ for every n. Let g be a limit point of $\{g_n\}$ in the compact set $C_{F,1}$. It is easy to verify that a series $\sum_{i=1}^{\infty} \alpha_i u_i$ converges iff $\sum_{i=1}^{\infty} g(|\alpha_i|) < \infty$, i.e. that g is equivalent to f.

We prove now the converse. Assume that $f \in C_{F,1}$. Since the extreme points of $C_{F,1}$ are contained in the compact set $E_{F,1}$ we get, by the Krein-Milman theorem, that

$$f(x) = \int_{E_{F,1}} \phi(\omega, x) d\mu(\omega) \qquad 0 \leqq x \leqq 1$$

for some probability measure μ on $E_{F,1}$. Set

$$E_{F,t} = \overline{\{F(sx)/F(s)\}_{0<s\leqq t}}, \qquad 0 < t < 1; E_F = \bigcap_{t>0} E_{F,t}$$

and let $\beta = \mu(E_F)$. Then $f(x) = \beta g(x) + (1-\beta)h(x)$ where $g \in C_F = \overline{\text{conv}}\, E_F$ and

$$h(x) = \int_0^1 F(sx)/F(s) d\nu(s)$$

for some probability measure ν on $[0,1]$ with $\nu(\{0\}) = 0$. (If $\beta = 0$, respectively 1, then g respectively h do not appear.) The argument used in the proof of Theorem 1 of [4] shows that there exists a normalized block basis $\{v_j\}$ of the unit vector basis $\{e_i\}$ of l_F such that a series $\sum_{j=1}^{\infty} \alpha_j v_j$ converges iff $\sum_{j=1}^{\infty} g(|\alpha_j|) < \infty$.

Since F satisfies the Δ_2 condition it follows immediately that $h(x)$ is equivalent to the Orlicz function $h_0(x)$ defined by

$$h_0(x) = \sum_{k=0}^{\infty} \nu_k F(2^{-k}x)/F(2^{-k}); \qquad \nu_k = \nu[2^{-(k+1)}, 2^{-k}].$$

Put $\lambda_k = \nu_k / F(2^{-k})$, $k = 1,2,\cdots$, and let k_0 be the smallest index such that $\lambda_{k_0} \neq 0$. Let σ be the set of all integers k for which $\lambda_{k_0} \leqq \lambda_k$. Then

$$\sum_{k\in\sigma} \lambda_k F(2^{-k}x) \leqq h_0(x) \leqq \sum_{k\in\sigma} \lambda_k F(2^{-k}x) + \lambda_{k_0} \sum_{k=k_0+1}^{\infty} F(2^{-k}x) \leqq$$

$$\leqq \sum_{k\in\sigma} \lambda_k F(2^{-k}x) + \lambda_{k_0} F(2^{-k_0}x) \sum_{j=1}^{\infty} 2^{-j} \leqq 2 \sum_{k\in\sigma} \lambda_k F(2^{-k}x).$$

* We assume, as we may without loss of generality, that $\lambda_i \geqq 0$ for all i.

(Recall that, by convexity, $F(\alpha y) \leqq \alpha F(y)$ whenever $0 \leqq \alpha,\ y \leqq 1$.) Since the λ_k for $k \in \sigma$ are bounded from below by the positive number λ_{k_0} it follows (by multiplying by $\lambda_{k_0}^{-1}$ and replacing each coefficient λ_k/λ_{k_0} by the integer $[\lambda_k/\lambda_{k_0}]$) that $h_0(x)$ is equivalent to a function $h_1(x) = \sum_{k=0}^{\infty} n_k F(2^{-k}x)$ with $\{n_k\}$ all integers.

Let now $\{A_j\}_{j=1}^{\infty}$ be disjoint infinite sets of integers and let $\{A_{j,k}\}_{k=0}^{\infty}$ be a disjoint partition of A_j so that $A_{j,k}$ has exactly n_k elements $(j,k = 1,2,\cdots,)$. Put

$$w_j = \sum_{k=0}^{\infty} 2^{-k} \sum_{i \in A_{j,k}} e_i \qquad j = 1,2,\cdots,$$

where, as before, $\{e_i\}$ denotes the unit vector basis in l_F. Notice that the series defining w_j converges in l_F since $\sum_{k=0}^{\infty} n_k F(2^{-k}) = h_1(1) < \infty$. Moreover, the same argument shows that a series $\sum_{j=1}^{\infty} \alpha_j w_j$ converges iff $\sum_{j=1}^{\infty} h_1(|\alpha_j|) < \infty$, i.e. iff $\sum_{j=1}^{\infty} h(|\alpha_j|) < \infty$.

Returning to the vectors v_j defined in the beginning of the proof in connection with the function g we may clearly assume that the support of v_j (i.e. the set of i such that e_i appears in the representation of v_j in terms of the unit vector basis) is disjoint from the support of w_m for all j and m. If we assume this, then $\sum_{j=1}^{\infty} \alpha_j(v_j + w_j)$ converges in l_F iff $\sum_{j=1}^{\infty} \alpha_j v_j$ and $\sum_{j=1}^{\infty} \alpha_j w_j$ both converge, that is iff $\sum_{j=1}^{\infty} g(|\alpha_j|) < \infty$ and $\sum_{j=1}^{\infty} h(|\alpha_j|) < \infty$. In other words $\sum_{j=1}^{\infty} \alpha_j(v_j + w_j)$ converges iff $\sum_{j=1}^{\infty} f(|\alpha_j|) < \infty$ and thus the closed linear span of $\{v_j + w_j\}_{j=1}^{\infty}$ is isomorphic to l_f. Q.E.D.

The dual version of Theorem 1 gives a characterization of the Orlicz sequence spaces l_f which are isomorphic to quotient spaces of l_F. Before stating this reformulation of Theorem 1, let us recall some known facts concerning duality in Orlicz sequence spaces (a general reference for these facts is [2] or [3]). If l_F is a separable Orlicz sequence space (i.e. if F satisfies Δ_2) then the dual space l_F^* is isomorphic to another Orlicz sequence space which we denote by l_{F^*}. Here F^* is defined by $F^*(s) = \sup\{ts - F(t), 0 < t < \infty\}$ and is called the complementary Orlicz function to F. (The definition of F^* requires F to be defined for all $t > 0$. If F is defined only in a neighborhood of 0 we extend it to an Orlicz function F_1 defined for all $t > 0$. The special choice of F_1 does not change the equivalence class of F^*.) The Orlicz function F^* satisfies the Δ_2 condition (at 0) iff l_F is reflexive and this is the case iff $\liminf_{x \to 0} xF'(x)/F(x) > 1$ (in addition to the standing assumption that F itself satisfies Δ_2).

COROLLARY. *Let l_F be a reflexive Orlicz sequence space. Then l_f is isomorphic to a quotient space of l_F if and only if f^* is equivalent to a function in $C_{F^*,1}$.*

This is an immediate consequence of Theorem 1. Let us remark that if l_F is not reflexive (but separable) then l_F contains a complemented subspace isomorphic to l_1 (this follows from a general result of R. C. James on spaces with unconditional bases, cf. also [3]). Hence, in this case any separable Banach space is isomorphic to a quotient space of l_F.

It follows immediately from the definition of F^* that for $a, b > 0$ $[aF(bt)]^*(s) = aF^*(s/ab)$, and thus $[F(xt)/F(x)]^*(s) = F^*(sF(x)/x)/F(x)$, $0 < x, s$. It is easily checked that if l_F is reflexive then $F(x)/x \to 0$ as $x \to 0$ while $F^*(F(x)/x)/F(x)$ remains bounded and bounded away from 0. Hence, if l_F is reflexive then f is equivalent to a function in $E_{F,1}$ iff f^* is equivalent to a function in $E_{F^*,1}$. In other words, (putting, for a set of functions A, $A^* = \{f^*; f \in A\}$) we get that, up to equivalence, $E_{F^*,1}$ is equal to $E^*_{F,1}$ and similarly E_{F^*} is equal to E^*_F. On the other hand, examples 1 and 2 of Section 3 show that in general $C_{F^*,1}$ respectively C_{F^*} are different from $C^*_{F,1}$ respectively C^*_F. This observation has a bearing on the question of complemented subspaces of l_F to which we turn next.

From Theorem 1 and its Corollary we deduce that if l_F is reflexive, a necessary condition on f for l_f to be isomorphic to a complemented subspace of l_F is

(*) *f is equivalent to a function in $C_{F,1}$ and also to a function in $C^*_{F^*,1}$.*

It is also easy to give a sufficient condition on f for l_f to be isomorphic to a complemented subspace of l_F. It is well known (cf. e.g. [5]) that if a Banach space X has a symmetric basis $\{e_i\}$ then for every block basis $\{u_n\}$ of the form $u_n = \alpha_n \sum_{i=p_n+1}^{p_{n+1}} e_i$, there is a projection (called averaging projection or conditional expectation) from X onto the closed linear span of the $\{u_n\}$. In an Orlicz sequence space l_F with unit vectors $\{e_i\}$, normalized blocks $\{u_n\}$ as above correspond (via the general correspondence between blocks and functions in $C_{F,1}$ which has been used in the proof of Theorem 1) to functions in $E_{F,1}$. Hence (as observed already in [3]) l_f is isomorphic to a complemented subspace of l_F if the following holds

$(^*_*)$ *f is equivalent to a function in $E_{F,1}$.*

It follows from the discussion above that for reflexive spaces l_F $(^*_*) \Rightarrow (*)$. This implication follows also trivially directly since $C^*_{F^*,1} \supset E^*_{F^*,1}$ and the latter set, is up to equivalence equal to $E_{F,1}$.

In some cases (cf. example 1 in Section 3 below) it happens that (*) is equivalent to $(^*_*)$. In such cases each of (*) and $(^*_*)$ give a necessary and sufficient condition on f for l_f to be isomorphic to a complemented subspace of l_F. In general

($\overset{*}{*}$) is strictly stronger than (*). In Section 3 we present an example (example 4) in which (*) is not sufficient to ensure that l_f is isomorphic to a complemented subspace of l_F. In general it is also quite difficult to check for which f condition (*) is satisfied. We turn, therefore, our attention to condition ($\overset{*}{*}$) which is often much simpler to check. We do not know whether ($\overset{*}{*}$) is also a necessary condition. The main result of this section shows that a somewhat weaker version of ($\overset{*}{*}$) is already necessary. Let us first write down the negation of ($\overset{*}{*}$) explicitly. Since $E_{F,1}$ is a compact set it follows that f is not equivalent to any function in $E_{F,1}$ iff

(+) *For every* $K \geqq 1$ *there exist* m_K *points* $x_i \in (0,1)$; $i = 1,2,\cdots,m_K$, *such that for every* $s \in (0,1)$ *there is at least one index* i, $1 \leqq i \leqq m_K$, *for which either* $F(sx_i)/F(s) < K^{-1}f(x_i)$ *or* $F(sx_i)/F(s) > Kf(x_i)$.

DEFINITION 1. *Let F be an Orlicz function satisfying the Δ_2 condition. A function f on $(0,1)$ is said to be strongly non-equivalent to $E_{F,1}$ if (+) holds with the additional requirement that the integers m_K can be chosen so that $m_K = o(K^\alpha)$ as $K \to \infty$ for every $\alpha > 0$.*

In terms of this notion of strong non-equivalence we can now consider the variant of ($\overset{*}{*}$) which gives a necessary condition for the existence of a projection from l_F on a subspace isomorphic to l_f.

THEOREM 2. *Let F be an Orlicz function satisfying the Δ_2 condition and let f be an Orlicz function strongly non-equivalent to $E_{F,1}$. Then l_f is not isomorphic to a complemented subspace of l_F.*

PROOF. Let $\{e_j\}$ be the unit vector basis of l_F. The same argument as the one used in the beginning of the proof of Theorem 1 shows that if l_f is isomorphic to a complemented subspace of l_F then the following holds: there exists a block basic sequence $w_i = \sum_{j\in\sigma_i}\lambda_j e_j$, $\lambda_j \geqq 0$, $\| w_i \| = 1$, $i = 1,2,\cdots$, (where σ_i are mutually disjoint finite sets of integers) so that if we put $g_i(x) = \sum_{j\in\sigma_i} F(\lambda_j x)$ then $|g_i(x) - g(x)| \leqq 2^{-i}$, $i = 1,2,\cdots$; $0 \leqq x \leqq 1$ for some $g \in C_{F,1}$ which is equivalent to f, and so that there is a projection P_1 from l_F onto $X = \overline{\text{span}}\,\{w_i\}_{i=1}^\infty$.

Since F satisfies the Δ_2 condition there is a p such that $xF'(x)/F(x) \leqq p$ for all $0 \leqq x \leqq 1$. It follows that for every $G \in C_{F,1}$ we have also that $xG'(x)/G(x) \leqq p$ and thus by integrating from x to ax with $1 < a < 1/x$ we get that

$$G(ax)/G(x) \leqq a^p, \ G \in C_{F,1}, \ 1 \leqq a \leqq 1/x.$$

Modifying the values of F for $z > 1$ by setting $F(z) = 1/F(z^{-1})$; $z > 1$ we can easily show that

$$G(ax)/G(x) \leqq a^p; \; G \in C_{F,1};$$

for any $a > 1$ and $x > 0$.

Since f and therefore g is strongly non-equivalent to $E_{F,1}$, we can choose a number K and m_K points $x_r \in (0,1)$; $r = 1,2,\cdots,m_K$ such that

$$m_K/K^{1/p} \leqq \min(2^{-1}4^{-p^{-1}} \| P_1 \|^{-1}, 2^{-(p+1)} \| P_1 \|^{-p})$$

and having the property that for every $s \in (0,1)$ there exists at least one r; $1 \leqq r \leqq m_k$ for which either $F(sx_r)/F(s) < K^{-1}g(x_r)$ or $F(sx_r)/F(s) > Kg(x_r)$.

Denote $\varepsilon = \min_{1 \leqq r \leqq m_k}\{g(x_r)\}$ and let i_ε be an integer for which $2^{-i} < \varepsilon$ if $i \geqq i_\varepsilon$. In the remainder of the proof we will concentrate our attention on the vectors $w_i = \sum_{j \in \sigma_i} \lambda_j e_j$; $i \geqq i_\varepsilon$ and on the Orlicz functions $g_i(x)$ with $i \geqq i_\varepsilon$ for which $|g_i(x) - g(x)| < 2^{-i} < \varepsilon$; $0 \leqq x \leqq 1$.

Let R be a projection of norm 1 from X onto the closed subspace X_0 generated by w_i; $i \geqq i_\varepsilon$. Obviously, $P = RP_1$ is a projection of norm $\| P \| \leqq \| P_1 \|$ from the whole space l_F onto X_0.

Now let us split the set σ_i into $2m_K$ disjoint subsets $\delta_i^{(r)}$ and $\eta_i^{(r)}$; $r = 1,2,\cdots,m_K$; $i \geqq i_\varepsilon$, having the property:

$$\left.\begin{array}{l} F(\lambda_j x_r)/F(\lambda_j) < K^{-1}g(x_r); \; j \in \delta_i^{(r)} \\ F(\lambda_j x_r)/F(\lambda_j) > Kg(x_r); \; j \in \eta_i^{(r)} \end{array}\right\} r = 1,2,\cdots,m_K; \; i \geqq i_\varepsilon$$

We have that

$$Kg(x_r) \sum_{j \in \eta_i^{(r)}} F(\lambda_j) \leqq \sum_{j \in \eta_i^{(r)}} F(\lambda_j x_r) \leqq g_i(x_r) < g(x_r) + \varepsilon < 2g(x_r)$$

which implies $\sum_{j \in \eta_i^{(r)}} F(\lambda_j) \leqq 2/K$ and thus

$$\sum_{r=1}^{m_K} \sum_{j \in \eta^{(r)}} F(\lambda_j) \leqq 2\, m_K/K; \; i \geqq i_\varepsilon.$$

Notice that if we set $h_i(x) = \sum_{r=1}^{m_K} \sum_{j \in \eta_i}^{(r)} F(\lambda_j x)$; $i \geqq i_\varepsilon$ then $h_i(x)/h_i(1) \in C_{F,1}$; $i \geqq i_\varepsilon$ and thus, as we have shown before

$$h_i(2\| P \|) \leqq 2^p \| P \|^p h_i(1) \leqq 2^{p+1} \| P_1 \|^p m_K/K; \; i \geqq i_\varepsilon.$$

The inequality satisfied by the ratio m_K/K leads to $h_i(2\| P \|) \leqq 1$; $i \geqq i_\varepsilon$, which means that the vectors $v_i = \sum_{r=1}^{m_K} \sum_{j \in \eta_i^{(r)}} \lambda_j e_j$; $i \geqq i_\varepsilon$ have norms $\leqq 1/(2\| P \|)$.

Denote $u_i^{(r)} = \sum_{j \in \delta_i^{(r)}} \lambda_j e_j$; $r = 1,2,\cdots,m_K$; $i \geqq i_\varepsilon$ and let Q_i be the projection of norm 1 from l_F onto the closed subspace generated by the vectors e_j; $j \in \sigma_i$. Then

$$w_i = Pw_i = P\left(\sum_{r=1}^{m_K} u_i^{(r)} + v_i\right) = \sum_{r=1}^{m_K} Pu_i^{(r)} + Pv_i$$

and further

$$w_i = \sum_{r=1}^{m_K} Q_iPu_i^{(r)} + Q_iPv_i;\ i \geqq i_\varepsilon.$$

This implies that:

$$\sum_{r=1}^{m_K} \| Q_iPu_i^{(r)} \| \geqq \| w_i \| - \| Q_iPv_i \| \geqq 1 - \| P \| \cdot \| v_i \| \geqq \tfrac{1}{2};\ i \geqq i_\varepsilon.$$

Hence, for every $i \geqq i_\varepsilon$ there exists at least one index r_i; $1 \leqq r_i \leqq m_K$, such that

$$\| Q_iPu_i^{(r_i)} \| \geqq 1/2\, m_K\ \ i \geqq i_\varepsilon.$$

Define the scalars $c_n^{(i)}$; $i, n \geqq i_\varepsilon$ by $Pu_i^{(r_i)} = \sum_{n=i_\varepsilon}^{\infty} c_n^{(i)} w_n$. As is well known (cf. e.g. [7]) the operator D (for diagonal) defined by

$$D \sum_{i=i_\varepsilon}^{\infty} \alpha_i u_i^{(r_i)} = \sum_{i=i_\varepsilon}^{\infty} \alpha_i c_i^{(i)} w_i$$

whenever $\sum_{i=i_\varepsilon}^{\infty} \alpha_i u_i^{(r_i)}$ converges, is bounded and $\| D \| \leqq \| P \|$.

Notice that

$$| c_i^{(i)} | = \| Q_iPu_i^{(r_i)} \| \geqq 1/2m_K;\ i \geqq i_\varepsilon$$

which implies that

$$\left\| \sum_{i=i_\varepsilon}^{q} \alpha_i w_i \right\| \leqq 2m_K \left\| \sum_{i=i_\varepsilon}^{q} \alpha_i c_i^{(i)} w_i \right\| \leqq 2m_K \| P \| \left\| \sum_{i=i_\varepsilon}^{q} \alpha_i u_i^{(r_i)} \right\|$$

for every finite set of scalars α_i; $i = i_\varepsilon, i_\varepsilon + 1, \cdots, q$. Choosing $q \geqq i_\varepsilon$ such that $2 \leqq \sum_{i=i_\varepsilon}^{q} g(x_{r_i}) \leqq 3$ we obtain that

$$1 \leqq \sum_{i=i_\varepsilon}^{q} g_i(x_{r_i}) \leqq 4$$

which means that $\| \sum_{i=i_\varepsilon}^{q} x_{r_i} w_i \| \geqq 1$. Using the previous inequality for $\alpha_i = x_{r_i}$ we have

$$\left\| \sum_{i=i_\varepsilon}^{q} x_{r_i} u_i^{(r_i)} \right\| \geqq 1/(2m_K \| P \|).$$

In order to get another estimate for $\| \sum_{i=i_\varepsilon}^{q} x_{r_i} u_i^{(r_i)} \|$ we should notice that

$$\sum_{i=i_\varepsilon}^{q} \sum_{j \in \delta_i^{(r_i)}} F(\lambda_j x_{r_i}) \leqq K^{-1} \sum_{i=i_\varepsilon}^{q} g(x_{r_i}) \sum_{j \in \delta_i^{(r_i)}} F(\lambda_j) \leqq 3K^{-1}$$

Since the function $\sum_{i=i_c} \sum_{j\in\delta_i^{(r_i)}} F(\lambda_j x_{r_i} x)$, normalized so that it gets the value one for $x = 1$, belongs to $C_{F,1}$ it follows that

$$\left\| \sum_{i=i_c}^{q} x_{r_i} u_i^{(r_i)} \right\| \leqq 3^{1/p} K^{-1/p},$$

and thus $1/2m_K \| P_1 \| \leqq 3^{1/p} K^{-1/p}$, which contradicts the choice of K and m_K.

Q.E.D.

3. Examples and special classes of Orlicz sequence spaces

We start by examining examples of Orlicz sequence spaces l_F in which F is given by an explicit expression. We consider first the most widely used examples of Orlicz spaces (besides, of course, the l_p spaces).

EXAMPLE 1. Let $F(x) = x^p(-\log x)^\alpha$ with $1 < p < \infty$ and $\alpha > 0$. It is easily checked that this function is an Orlicz function on some interval $0 \leqq x < x_0 < 1$ and thus can be extended to an Orlicz function on $(0, \infty)$. Since for all our discussion the values of an Orlicz function outside a neighbourhood of 0 are of no importance, we define it explicitely only in a suitable neighborhood of 0. (This is called the "principal part" of an Orlicz function.) We have that

$$\lim_{t\to 0} \frac{F(xt)}{F(t)} = \lim_{t\to 0} x^p\left(1 + \frac{\log x}{\log t}\right)^\alpha = x^p$$

and thus E_F consists only of the function x^p, while $E_{F,1}$ consists of two equivalence classes: x^p and functions equivalent to $F(x)$. Also $C_{F,1}$ contains only these two equivalence classes. Indeed, any function in $C_{F,1}$ is equivalent to a function in C_{F,x_0} and C_{F,x_0} consists exactly of functions of the form $f(x) = \int_0^{x_0} F(xt)/F(t) d\mu(t)$ for some probability measure μ on $[0, x_0]$. (In this integral $F(x\cdot 0)/F(0)$ stands for x^p). For a fixed $0 \leqq x \leqq 1$, $F(xt)/F(t)$ is an increasing function of t. Hence for such x, $F(x_0 x)/F(x_0) \geqq f(x) \geqq \mu[x_1, x_0] F(x_1 x)/F(x_1)$ for every $x_1 < x_0$. Thus, unless μ is concentrated in the origin, $f(x)$ is equivalent to $F(x)$. It follows that the only Orlicz sequence spaces which are isomorphic to subspaces of l_F, are l_F itself and l_p, and both are obviously also isomorphic to complemented subspaces of l_F. In order to find the Orlicz sequence spaces which are quotient spaces of l_F we pass to the complementary function F^* of F. It is easy to show (cf. e.g. [2]) that F^* is equivalent to a function whose principal part is $G(x) = x^q |\log x|^{-\alpha(q-1)}$ where $q^{-1} + p^{-1} = 1$. E_G and C_G consist only of x^q. However $C_{G,1}$ turns out to contain infinitely many equivalent classes. Using, for example, the fact that for $0 < \varepsilon < \beta = \alpha(q-1)$

$$\int_1^\infty \left(\frac{u}{u+y}\right)^\beta \frac{du}{u^{1+\varepsilon}} = y^{-\varepsilon} \int_{1/y}^\infty \left(\frac{v}{1+v}\right)^\beta \frac{dv}{v^{1+\varepsilon}} = Cy^{-\varepsilon} + 0(y^{-\beta}) \text{ as } y \to \infty$$

we get that

$$H_\varepsilon(x) = \int_0^{e^{-1}} \frac{G(sx)}{G(s)} \frac{ds}{s|\log s|^{1+\varepsilon}} = x^q \int_0^{e^{-1}} \left(\frac{\log s}{\log x + \log s}\right)^\beta \frac{ds}{s|\log s|^{1+\varepsilon}}$$

$$= x^q \int_1^\infty \left(\frac{u}{u+|\log x|}\right)^\beta \frac{du}{u^{1+\varepsilon}}$$

is equivalent to $x^q(-\log x)^\varepsilon$. Clearly $H_\varepsilon(x)/H_\varepsilon(1) \in C_{G,1}$. It follows that l_F has many Orlicz sequence spaces as quotient spaces e.g. all the space $l_{x^p(-\log x)^\gamma}$ with $0 \leqq \gamma \leqq \alpha$. It is also clear from the discussion above that for any $g \in C_{G,1}$ such that g is not equivalent to a function in $E_{G,1}$ (i.e. to G itself or x^q) l_g is isomorphic to a subspace of l_G but not to a complemented subspace of l_G. This example shows among other things how much different $C_{F,1}$ and $C^*_{F^*,1}$ can be.

We turn next to an example defined and investigated by Lindberg [3] which we would like to investigate here a little further.

EXAMPLE 2. Let $F(x) = x^{p+\sin(\log(-\log x))}$. A simple computation shows that if $p > 1 + \sqrt{2}$ then $F(x)$ is an Orlicz function in some interval $[0, x_0]$ with $x_0 > 0$. Put

$$U(x) = xF'(x)/F(x) = p + \sin(\log(-\log x)) + \cos(\log(-\log x))$$

Then for any $\delta > 0$, $\lim_{t\to 0}(U(tx) - U(t)) = 0$ uniformly for x in $[\delta, 1]$. It follows easily that whenever $\{t_n\}$ is a sequence converging to 0 with $\lim_{n\to\infty} U(t_n) = s$ then $F(xt_n)/F(t_n)$ converges uniformly to x^s. All these observations are due to Lindberg [3].

The set E_F consists thus exactly of the functions x^s with s ranging over the interval $[p-\sqrt{2},\ p+\sqrt{2}]$ while the set C_F consists of all the functions $f(x)$ which can be represented as

$$f(x) = \int_{p-\sqrt{2}}^{p+\sqrt{2}} x^s d\mu(s); \qquad 0 \leqq x \leqq 1$$

for some probability measure μ on $[p-\sqrt{2}, p+\sqrt{2}]$. By taking, for example, μ to be uniformly distributed on $[t, p+\sqrt{2}]$ we get a function equivalent to $-x^t/\log x$.

A simple computation shows that if $f \in C_F$ then $\lim_{t \to 0} f(tx)/f(t) = x^s$ where is the smallest number in the support of the measure μ representing f. It follows sn particular that F is not equivalent to any function in C_F and thus according to Theorem 2 of [4], the space l_F has up to equivalence a unique symmetric basis (i.e. if l_F is isomorphic to some l_G then G must be equivalent to F).

This example shows also that in general C_F and $C^*_{F^*}$ are different (in Example 1 both sets consisted of the same single element). We shall show that if f is equivalent to a function in C_F and to a function in $C^*_{F^*}$ then f is already equivalent to a function in E_F, i.e. to x^s for some $s \in [p - \sqrt{2}, p + \sqrt{2}]$. Indeed, let $f \in C_F$ and let s be the smallest number in the support of the measure μ which represents f. If f is not equivalent to x^s then $\mu(\{s\}) = 0$ and hence $\lim_{x \to 0} f(x)/x^s = 0$ while $E_f = \{x^s\}$. Let $g \in C_{F^*}$ be such that g^* is equivalent to f.

Then on the one hand

$$\text{(i)} \qquad \lim_{y \to 0} g(y)/y^r = \infty, \quad E_g = \{y^r\}, \quad r^{-1} + s^{-1} = 1$$

and on the other hand for some measure ν on $[t_1, t_2]$ (where $t_1^{-1} + (p - \sqrt{2})^{-1} = 1$ and $t_2^{-1} + (p + \sqrt{2})^{-1} = 1$)

$$\text{(ii)} \qquad g(y) = \int_{t_1}^{t_2} y^t d\nu(t). \quad ; \ 0 \leqq y \leqq 1$$

Since (i) and (ii) are mutually contradictory, our assertion is proved.

EXAMPLE 3. We turn now to the example constructed in the proof of Theorem 3 of [4], which will play the central role in this section. Let us first recall the definition of the two Orlicz functions $M(x)$ and $N(x)$ used in this example. Let $1 < c < d < \infty$ be given, and let $t_n = 2^{-2^{n-1}}$, $n = 1, 2, \cdots$. The functions $M(x)$ and $N(x)$ are Orlicz functions (defined on $[0,1]$) which satisfy

(i) $M(1) = N(1) = 1$, $M'(1) = N'(1) = c$,

(ii) $c \leqq x\,M'(x)/M(x) \leqq d$, $c \leqq x\,N'(x)/N(x) \leqq d$, $0 < x \leqq 1$

(iii) $M(t_{3n+1})/N(t_{3n+1}) = \lambda^{2^{n-1}}$, for some $0 < \lambda < 1$, $n = 1, 2, \cdots$

and the following recursion relations ($n = 1, 2, \cdots$)

$$\left.\begin{aligned} M(x) &= M(t_{3n+1})N(x/t_{3n+1}) \\ N(x) &= N(t_{3n+1})N(x/t_{3n+1}) \end{aligned}\right\} t_{3n+1} \geqq x \geqq t_{3n+2} = t_{3n+1}^2$$

$$\text{(iv)} \qquad \left.\begin{aligned} M(x) &= M(t_{3n+2})M(x/t_{3n+2}) \\ N(x) &= N(t_{3n+2})N(x/t_{3n+2}) \end{aligned}\right\} t_{3n+2} \geqq x \geqq t_{3n+3} = t_{3n+2}^2$$

$$\left.\begin{aligned} M(x) &= M(t_{3n+3})M(x/t_{3n+3}) \\ N(x) &= N(t_{3n+3})M(x/t_{3n+3}) \end{aligned}\right\} t_{3n+3} \geqq x \geqq t_{3n+4} = t_{3n+3}^2$$

It is easily seen that such M and N exist (cf [4]). We begin by establishing a further property of these functions.

LEMMA 1. *For any integer n set $j_n = 2^{3(n+1)}$. Then for any non-negative integer k at least one of the three functions*

$$M(2^{-i}x)/M(2^{-i}),\ i = kj_n,\ kj_n + j_{n-1},\ kj_n + 4j_{n-1},$$

is equal to $M(x)$ for $1 \geqq x \geqq t_{3n+1}$ and at least one of these functions is equal to $N(x)$ for $1 \geqq x \geqq t_{3n+1}$. The same statement is valid for the functions

$$N(2^{-i}x)/N(2^{-i}),\ i = kj_n,\ kj_n + j_{n-1},\ kj_n + 4j_{n-1}\ .$$

PROOF. We shall prove simultaneously all the assertions in the lemma by induction on k (keeping n fixed). For $k = 0$ the assertions follow from the fact that

$$M(2^0x)/M(2^0) = M(x),\ N(2^0x)/N(2^0) = N(x),$$

and by the recursion relations (iv)

$$M(2^{-j_{n-1}}x)/M(2^{-j_{n-1}}) = M(t_{3n+1}x)/M(t_{3n+1}) = N(x),\ t_{3n+1} \leqq x \leqq 1\ ,$$

and

$$N(2^{-4j_{n-1}}x)/N(2^{-4j_{n-1}}) = N(t_{3n+3}x)/N(t_{3n+3}) = M(x), \quad t_{3n+1} \leqq x \leqq 1.$$

This proves the lemma for $k = 0$. To illustrate the inductive step let us prove the lemma for $k = 1$. Notice that for $j_n \leqq i \leqq 2j_n - j_{n-1}$ and for $t_{3n+1} \leqq x \leqq 1$ we have

$$t_{3n+4} = 2^{-j_n} \geqq 2^{-i}x \geqq 2^{-2j_n + j_{n-1}}t_{3n+1} = 2^{-2j_n} = t_{3n+4}^2.$$

Thus for such i and x

$$M(2^{-i}x)/M(2^{-i}) = M(t_{3n+4}2^{-r}x)/M(t_{3n+4}2^{-r}) = N(2^{-r}x)/N(2^{-r})$$

where $r = i - j_n$. A similar equation holds for $N(2^{-i}x)/N(2^{-i})$. This reduces the case $k = 1$ to the case $k = 0$.

Suppose now the assertion of the lemma is true for $k < r$ and let us prove it for $k = r > 1$. Choose the integer m so that $2^{m-1} \leqq rj_n < 2^m$, and let $s = 2^{m-1}/j_n$ which is an integer (a positive power of 2). For $rj_n \leqq i \leqq (r+1)j_n - j_{n-1}$ and $t_{3n+1} \leqq x \leqq 1$ we have that $t_m = 2^{-2^{m-1}} \geqq 2^{-i}x \geqq 2^{-(r+1)j_n + j_{n-1}}\ t_{3n+1} \geqq 2^{-2^m} =$

$= t_m^2$. Hence, by the recursion relations (iv), we get that for these i and x, $M(2^{-i}x)/M(2^{-i})$ is equal to either (depending on m (mod 3))

$$M(2^{-i+sj_n}x)/M(2^{-i+sj_n}) \text{ or } N(2^{-i+sj_n}x)/N(2^{-i+sj_n}).$$

A similar statement holds for $N(2^{-i}x)/N(2^{-i})$. Thus we are reduced to the case $k = r - s$ for which the lemma holds by the induction hypothesis.

THEOREM 3. *There exists a reflexive Orlicz sequence space which has no complemented subspace isomorphic to an l_p space, $1 < p < \infty$.*

PROOF. We claim that the space l_M with M the function considered above has the desired property. By Theorem 2 it is enough to show that for every $1 < p < \infty$ the function x^p is strongly non-equivalent to $E_{M,1}$.

Let $p > 1$ be given, let n be an integer and let $K = d_0^{-1}\lambda^{-2n-4}$ where $d_0 = 2^d$ while d and λ $(0 < \lambda < 1)$ are the constants appearing in (ii) and (iii) above.

Consider now the $17 \cdot 2^{3n}$ points 2^{-k}; $k = 1,2,\cdots,17.2^{3n}$ and assume there exists $s \in (0,1)$ such that

$$K^{-1}2^{-kp} \leqq M(s2^{-k})/M(s) \leqq K2^{-kp}; \qquad k = 1,2,\cdots,17\cdot 2^{3n}.$$

Let the integer i satisfy the inequality $2^{-(i+1)} < s \leqq 2^{-i}$. Using (ii) i.e. the Δ_2 condition for M, we get

$$d_0^{-1}M(2^{-i}2^{-k})/M(2^{-i}) \leqq M(s2^{-k})/M(s) \leqq d_0 M(2^{-i}2^{-k})/M(2^{-i}).$$

It follows that

$$(d_0K)^{-1}2^{-kp} \leqq M(2^{-i}2^{-k})/M(2^{-i}) \leqq d_0K2^{-kp}; \qquad k = 1,2,\cdots,17\cdot 2^{3n}.$$

These inequalities can be rewritten as follows:

$$(d_0K)^{-1}2^{-jp} \leqq M(2^{-i}\cdot 2^{-j})/M(2^{-i}) \leqq (d_0K)2^{-jp}; \qquad j = 1,2,\cdots, 16\cdot 2^{3n}$$

and

$$(d_0K)^{-1}2^{-(j+2^{3n})p} \leqq M(2^{-i}\cdot 2^{-j}\cdot 2^{-2^{3n}})/M(2^{-i}) \leqq (d_0K)2^{-(j+2^{3n})p}$$

$$j = 1,2,\cdots,16\cdot 2^{3n}.$$

Dividing the corresponding inequalities and noticing that $2^{-2^{3n}} = t_{3n+1}$ we have

$$(d_0K)^{-2}t_{3n+1}^p \leqq \frac{M(2^{-(i+j)}\, t_{3n+1})}{M(2^{-(i+j)})} \leqq (d_0K)^2 t_{3n+1}^p; \qquad j = 1,2,\cdots,16\cdot 2^{3n}.$$

Now, by applying Lemma 1 for the functions

$M(2^{-(i+j)}x)/M(2^{-(i+j)})$; $t_{3n+1} \leqq x \leqq 1$; $j=1,2,\cdots,16 \cdot 2^{3n} = 2j_n$, we find a pair of indices r_1 and r_2; $1 \leqq r_1, r_2 \leqq 16 \cdot 2^{3n}$ such that

$$\left.\begin{array}{l} M(2^{-(i+r_1)}x)/M(2^{-(i+r_1)}) = M(x) \\ M(2^{-(i+r_2)}x)/M(2^{-(i+r_2)}) = N(x) \end{array}\right\} ; \qquad t_{3n+1} \leqq x \leqq 1.$$

This means that

$$(d_0K)^{-2}\, t^p_{3n+1} \leqq M(t_{3n+1}) \leqq (d_0K)^2\, t^p_{3n+1},$$
$$(d_0K)^{-2}\, t^p_{3n+1} \leqq N(t_{3n+1}) \leqq (d_0K)^2\, t^p_{3n+1},$$

which implies that (cf (iii)),

$$\lambda^{-2^{n-1}} = \frac{N(t_{3n+1})}{M(t_{3n+1})} \leqq (d_0K)^4 = (\lambda^{-2^{n-4}})^4 = \lambda^{-2^{n-2}},$$

and this contradicts the fact that $\lambda < 1$. In conclusion, we have just shown that $M(x)$ and x^p; $p > 1$ satisfy the condition (+) of Section 2 with $K = d_0^{-1}\lambda^{-2^{n-4}}$ and $m_K = 17 \cdot 2^{3n}$. This means that for any $\alpha > 0$ $m_K = o(K^\alpha)$ as $K \to \infty$ and thus, x^p; $p > 1$ is strongly non-equivalent to $E_{M,1}$. Q.E.D.

If we look at the Orlicz sequence spaces from the point of view of the isomorphic theory of Banach spaces, it is quite natural to identify two Orlicz functions G and H provided $E_{G,1} = E_{H,1}$. The reason behind this statement is that if condition Δ_2 holds for G and H then $E_{G,1} = E_{H,1}$ implies that l_G is isomorphic to a complemented subspace of l_H and vice-versa l_H is isomorphic to a complemented subspace of l_G; therefore, by using Pelczynski's decomposition method [6] (which is also described in the last lines of the proof of [4] Theorem 3) it follows that l_G and l_H are isomorphic.

With this identification in mind we can introduce a partial order in the class of all Orlicz functions as follows: $G \prec H \Leftrightarrow G \in E_{H,1}$.

This leads us to the following definition.

DEFINITION 2. *An Orlicz function G is called minimal if $E_{G,1} = E_{H,1}$ for every $H \in E_{G,1}$.*

A standard application of Zorn's Lemma to the set of Orlicz functions in $E_{F,1}$ endowed with the previously introduced order proves that:

For every Orlicz function F on $[0,1]$ satisfying the Δ_2 condition there exists at least one minimal Orlicz function G in $E_{F,1}$.

Minimal Orlicz sequence spaces have the following interesting property.

THEOREM 4. *Let G be a minimal Orlicz function, and let $\{e_i\}$ be the unit vector basis of l_G. Then every block basis $\{u_k\}$ with respect to $\{e_i\}$ which has the form $u_k = \alpha_k \sum_{i \in \sigma_k} e_i$ (where $\{\sigma_k\}$ are mutually disjoint finite subsets of the integers and $\alpha_k \neq 0$, scalars) spans a subspace isomorphic to l_G itself.*

PROOF. Let $U = \overline{\text{span}}\{u_k\}$. We assume as we may, that $\| u_k \| = 1$ for every k. For $y = \sum_i \lambda_i e_i \in l_G$ we shall denote $\sum_i G(|\lambda_i|)$ by $\gamma(y)$. There is a contractive projection P from l_G onto U defined by

$$Py = \sum_{k=1}^{\infty} \left(\sum_{i \in \sigma_k} \lambda_i \right) u_k / \alpha_k n_k \text{ if } y = \sum_i \lambda_i e_i,$$

where n_k denotes the number of elements in σ_k. By the convexity of G

$$\gamma(Py) = \sum_{k=1}^{\infty} n_k G \left(\left| \sum_{i \in \sigma_k} \lambda_i \right| / n_k \right) \leqq \sum_{k=1}^{\infty} \sum_{i \in \sigma_k} G(|\lambda_i|) \leqq \gamma(y).$$

Let $G_k(t) = \gamma(t u_k) = n_k G(|\alpha_k| t)$, $0 \leqq t \leqq 1$. Then $G_k \in E_{G,1}$ and there is a subsequence $\{k_j\}$ of the integers such that $|G_{k_j}(t) - H(t)| \leqq 2^{-j}$, $0 \leqq t \leqq 1$, for some $H \in E_{G,1}$. The subspace $U_0 = \overline{\text{span}}\,\{u_{k_j}\}$ is isomorphic to l_H and there is a projection from U onto U_0. Since G is minimal, $G \in E_{H,1}$ and thus l_H has a complemented subspace isomorphic to l_G. We shall use these facts and a slight variant of Pelczynski's decomposition method [6] to show that $l_G \approx U$ ($\approx$ denotes isomorphism).

For a subspace W of l_G we define $(W \oplus W \oplus \cdots)_G$ as the space of all sequences $y = (y_1, y_2, \cdots)$ such that each vector $y_n \in W$ and $\Gamma(y) = \sum_{n=1}^{\infty} \gamma(y_n) < \infty$. The norm is defined as usual $\| y \| = \{\inf t > 0, \Gamma(y/t) \leqq 1\}$. Observe that in general $(W \oplus W \oplus \cdots)_G$ *does not* coincide with the space of all the $y = (y_1, y_2, \cdots)$ such that $(\| y_1 \|, \| y_2 \|, \cdots) \in l_G$.

Consider the subspace U of l_G, the projection P onto U and let $W =$ kernel P. Since G satisfies condition Δ_2 (with constant K say) we have for every choice of scalars λ_i and μ_i that

$$\sum_i G(|\lambda_i + \mu_i|) \leqq \sum_i G(2 \max(|\lambda_i|, |\mu_i|)) \leqq K \left(\sum_i G(|\lambda_i|) + \sum_i G(|\mu_i|) \right)$$

and hence $\gamma(y - Py) \leqq 2K\gamma(y)$ for every $y \in l_G$. It follows that

$$l_G \approx (l_G \oplus l_G \oplus \cdots)_G \approx ((U \oplus W) \oplus (U \oplus W) \oplus \cdots)_G$$

$$\approx U \oplus ((W \oplus U) \oplus (W \oplus U) \oplus \cdots)_G \approx U \oplus l_G.$$

Since we have also that $U \approx l_G \oplus X$ for some Banach space X we deduce that $U \approx l_G \oplus U \approx l_G$ as desired.

REMARK. M. Zippin [8] has shown that if $\{e_i\}$ is a basis of a Banach space such that for every normalized block basic sequence $\{u_k\}$ of the form appearing in the statement of the theorem, $\{e_i\}$ is equivalent to $\{u_i\}$ then $\{e_i\}$ is already equivalent to the unit vector basis of c_0 or l_p for some $1 \leqq p < \infty$. There are minimal Orlicz functions G which are not equivalent to any function x^p (see below). For such functions G, the isomorphism from l_G onto U is not in general induced by mapping the $\{e_i\}$ to the vectors $\{u_i\}$.

Let M be the Orlicz function of Example 3 above. Since by Theorem 3, $E_{M,1}$ does not contain a function equivalent to any x^p, it follows by the remark preceeding Theorem 4 that $E_{M,1}$ contains a minimal function G which is not equivalent to any x^p. It is perhaps worthwhile to note that M itself is equivalent to a minimal Orlicz function as is shown in the following Proposition.

PROPOSITION 1. *For any $G \in E_{M,1}$ there exists an Orlicz function M_1 equivalent to M such that $M_1 \in E_{G,1}$.*

PROOF. Let s_n; $0 < s_n \leqq 1$; $n = 1,2,\cdots$ be such that

$$\left| M(s_n x)/M(s_n) - G(x) \right| < G(t_{3n+6})/2; \; n = 1,2,\cdots; \; 0 \leqq x \leqq 1.$$

Choose integers i_n such that $2^{-(i_n+1)} < s_n \leqq 2^{-i_n}$; $n = 1,2,\cdots$ Then

$$d_0^{-1} M(2^{-i_n}x)/M(2^{-i_n}) \leqq M(s_n x)/M(s_n) \leqq d_0 M(2^{-i_n}x)/M(2^{-i_n}); \; n = 1,2,\cdots; \quad 0 \leqq x \leqq 1$$

which implies

$$d_0^{-1}\left(G(x) - \frac{G(t_{3n+6})}{2}\right) \leqq \frac{M(2^{-i_n}x)}{M(2^{-i_n})} \leqq d_0\left(G(x) + \frac{G(t_{3n+6})}{2}\right); \; n = 1,2,\cdots; \quad 0 \leqq x \leqq 1$$

and thus

$$(d_0^{-1}/2)G(x) \leqq M(2^{-i_n}x)/M(2^{-i_n}) \leqq (3d_0/2)G(x); \; n = 1,2,\cdots; \quad t_{3n+6} \leqq x \leqq 1.$$

Since $2^{-16 \cdot 2^{3n}} t_{3n+1} = 2^{-17 \; 2^{3n}} > 2^{-2^{3n+5}} = t_{3n+6}$ it follows that

$$(d_0^{-1}/2)G(2^{-j}) \leqq M(2^{-i_n}2^{-j})/M(2^{-i_n}) \leqq (3d_0/2)G(2^{-j}); \; n = 1,2,\cdots; \quad j = 1,2,\cdots,16.2^{3n}$$

and

$$(d_0^{-1}/2)G(2^{-j}z) \leqq M(2^{-i_n}2^{-j}z)/M(2^{-i_n}) \leqq (3d_0/2)G(2^{-j}z);\ n = 1,2,\cdots;$$
$$j = 1,2,\cdots,16 \cdot 2^{3n};\ t_{3n+1} \leqq z \leqq 1.$$

Dividing the corresponding inequalities we obtain

$$(d_0^{-2}/3)\ \frac{G(2^{-j}z)}{G(2^{-j})} \leqq \frac{M(2^{-(i_n+j)}z)}{M(2^{-(i_n+j)})} \leqq 3d_0^2\frac{G(2^{-j}z)}{G(2^{-j})};\ n = 1,2,\cdots; j=1,2,\cdots;\ 16.2^{3^n};$$
$$t_{3n+1} \leqq z \leqq 1$$

By Lemma 1 there exists an integer r_n; $1 \leqq r_n \leqq 2j_n$ such that

$$M(2^{-(i_n+r_n)}z)/M(2^{-(i_n+r_n)}) = M(z);\ n = 1,2,\cdots;\ t_{3n+1} \leqq z \leqq 1.$$

Hence

$$d_0^{-2}/3)G(2^{-r}\,z)/G(2^{-r_n}) \leqq M(z) \leqq 3d_0^2G(2^{-r_n}z)/G(2^{-r_n}); n=1,2,\cdots; t_{3n+1} \leqq z \leqq 1.$$

Since $G(2^{-r_n}x)/G(2^{-r_n})$ contains a subsequence converging uniformly on $[0,1]$ to a function $M_1 \in E_{G,1}$ it follows that

$$(d_0^{-2}/3)M_1(x) \leqq M(x) \leqq 3d_0^2M_1(x); \qquad 0 \leqq x \leqq 1$$

which means that M_1 is equivalent to M. Q.E.D.

Our next proposition shows that the collection of sets E_F with l_F reflexive is closed under unions (up to equivalence).

PROPOSITION 2. *Let F and G be Orlicz functions such that l_F and l_G are reflexive. Then there exists an Orlicz function H such that l_H is reflexive and with E_H equal up to equivalence to $E_{F,1} \cup E_{G,1}$. There exists also an Orlicz function $\tilde{H}$ with $l_{\tilde{H}}$ reflexive and $E_{\tilde{H}}$ equal, up to equivalence to $E_F \cup E_G$.*

PROOF. We assume that F and G are defined on $[0,1]$ and normalized so that $F(1) = G(1) = 1$. Since l_F and l_G are reflexive there is a $c > 1$ such that $xF'(x)/F(x) \geqq c$ and $xG'(x)/G(x) \geqq c$ for every $x \in [0,1]$. There is no loss of generality to assume that $F'(1) = G'(1) = c$. Indeed, we have simply to replace F by the equivalent function $F_1(x) = \max(F(x), c(x-1)+1)$, $0 \leqq x \leqq 1$ (and similarly for G).

Let $t^n = 2^{-2^{n-1}}$; $n = 1,2,\cdots$ and define $H(x)$ by

$$H(x) = \begin{cases} F(x) & t_1 = \tfrac{1}{2} \leqq x \leqq 1 \\ H(t_{2n-1})G(x/t_{2n-1}) & t_{2n} \leqq x \leqq t_{2n-1} \\ H(t_{2n})F(x/t_{2n}) & t_{2n+1} \leqq x \leqq t_{2n} \\ 0 & x = 0 \end{cases}$$

By Lemma 2 of [4] H is an Orlicz function on $[0,1]$ with l_H reflexive. Since $H(t_{2n-1}x)/H(t_{2n-1}) = G(x)$ for $t_{2n-1} \leqq x \leqq 1$ it follows that G belongs to E_H and thus $E_{G,1} \subset E_H$. Similarly $E_{F,1} \subset E_H$.

Let now $h(x) = \lim_m H(xs_m)/H(s_m) \in E_H$. Choose n_m so that $t_{n_m+1} < s_m \leqq t_{n_m}$ and let $u_m = s_m/t_{n_m+1}$. By passing to a subsequence we may assume that $u = \lim_m u_m$ exists (finite or infinite) and that e.g. all the n_m are even. If $u < \infty$ then the Δ_2 condition for H implies that $h(x)$ is equivalent to $\lim_m H(xt_{n_m+1})/H(t_{n_m+1}) = G(x)$. If $u = \infty$ then

$$H(xs_m)/H(s_m) = F(xs_m t_{n_m}^{-1})/F(s_m t_{n_m}^{-1});\ u_m^{-1} \leqq x \leqq 1$$

and hence $h \in E_{F,1}$. (If the n_m are odd then either h is equivalent to F or $h \in E_{G,1}$.)

To prove the second part we just write down the definition of $\tilde{H}$ and leave the details to the reader

$$\tilde{H}(x) = \begin{cases} \tilde{H}(t_{2n})F_n(x/t_{2n});\ t_{2n+1} \leqq x \leqq t_{2n} \\ \tilde{H}(t_{2n+1})G_n(x/t_{2n+1});\ t_{2n+2} \leqq x \leqq t_{2n+1} \end{cases} \quad n = 0,1,2,\cdots$$

where $\tilde{H}(1) = \tilde{H}(t_0) = 1$ and $F_n(x)$ is an Orlicz function on $[0,1]$ satisfying $F_n(1) = 1, F_n'(1) = c \leqq xF_n'(x)/F(x), 0 \leqq x \leqq 1$, and $K^{-1} \leqq F_n(x)F(2^{-n})/F(2^{-n}x) \leqq K$ for some K independent of x and n ($G_n(x)$ satisfies the same requirements only with F replaced by G).

COROLLARY. *Let l_F and l_G be reflexive Orlicz sequence spaces. Then there exists a reflexive Orlicz sequence space which contains a complemented subspace isomorphic to $l_F \oplus l_G$.*

PROOF. Let H be the function constructed in the previous proposition. Then l_F and l_G are both isomorphic to complemented subspaces of l_H, and the result follows from the fact that $l_H \oplus l_H \approx l_H$.

REMARK. We did not check whether Proposition 2 and its Corollary hold without the reflexivity assumption.

EXAMPLE 4. Let $1 < p < r < \infty$ and consider the function $H(x)$ obtained by applying Proposition 2 to $F(x)=x^p$ and $G_0(x)=x^r$. Here we can take p as the c appearing in the proof of Proposition 2. In order to use the definition of H given there we have to replace $G_0(x)$ by $G(x) = \max(x^r, p(x-1)+1)$, $0 \leqq x \leqq 1$. Up to equivalence, E_H consists only of two functions, x^p and x^r. The closed convex hull C_H of E_H contains functions equivalent to x^s for every $s \in [p, r]$. Indeed, for $1 \leqq u < \infty$ define $h_u(x) = \lim_n H(ut_{2n-1}x)/H(ut_{2n-1}) \in E_H$. Then

$$h_u(x) = \begin{cases} x^p & u^{-1} \leqq x \leqq 1 \\ \\ u^{-p}G(ux) & 0 \leqq x \leqq u^{-1} \end{cases} = \begin{cases} x^p & u^{-1} \leqq x \leqq 1 \\ u^{-p}(p(ux-1)+1) & x_0u^{-1} \leqq x \leqq u^{-1} \\ u^{r-p}x^r & 0 \leqq x \leqq x_0u^{-1} \end{cases}$$

where x_0 is defined by $x_0^r = p(x_0 - 1) + 1$, $0 < x_0 < 1$. For $1 < \alpha < r - p + 1$ consider

$$f_\alpha(x) = \int_1^\infty h_u(x)u^{-\alpha}du$$

$$= \int_1^{x_0/x} u^{r-p-\alpha}x^r du + \int_{x_0/x}^{1/x} u^{-p-\alpha}(p(ux-1)+1)du + \int_{1/x}^\infty x^p u^{-\alpha}du.$$

Then $f_\alpha(x)/f_\alpha(1) \in C_H$ and $f_\alpha(x) = K_\alpha x^{p+\alpha-1} + o(x^{p+\alpha-1})$ as $x \to 0$, for a suitable positive constant K_α, so that $f_\alpha(x)$ is equivalent to $x^{p+\alpha-1}$. A similar computation shows that for $r' \leqq s' \leqq p'$ (where $1/r' + 1/r = 1$ and $1/p' + 1/p = 1$) $x^{s'}$ is equivalent to a function in C_{H^*} where H^* is the complementary function to H. Thus we have that $s \in [p, r] \Leftrightarrow l_s$ is isomorphic to a subspace of $l_H \Leftrightarrow l_s$ is isomorphic to a quotient space of l_H. On the other hand l_s is isomorphic to a complemented subspace of l_H iff $s = p$ or $s = r$. In order to prove this assertion it is enough (by Theorem 2) to show that for $p < s < r$, the function x^s is strongly non-equivalent to $E_{H,1}$.

Let n be an integer and let $K = 2^{-\alpha \cdot 2^{n-1}}$ where $\alpha = \min(r - s, s - p) > 0$. A simple verification shows that for every $i > 2^{n+1}$ there is a $j \leqq 2^{n+1}$ such that the number $H(2^{-i}2^{-j})/H(2^{-i}) \cdot 2^{-js}$ is outside the interval $[K^{-1}, K]$. Put $x_j = 2^{-j}$; $j = 1, \cdots, 2^{n+1}$ and for $2^{n+1} < j \leqq 2^{n+2}$ choose x_j so that

$$H(2^{-j+2^{n+1}}x_j)/H(2^{-j+2^{n+1}})x_j^s$$

is outside $[K^{-1}, K]$. Then for every $0 < t \leqq 1$ there is a $1 \leqq j \leqq 2^{n+2}$ such that $H(tx_j)/H(t)x_j^s$ is outside $[K^{-1}2^d, K2^{-d}]$ where d is the Δ_2 constant of H (actually $d = r$). This proves that x^s is strongly non-equivalent to $E_{H,1}$.

Proposition 2 can be generalized to the case where instead of the union of two sets of the form E_F we consider suitable infinite unions. We shall state here only one such generalization which shows that the class of Orlicz functions, with the order defined above, has also relative maximal elements.

PROPOSITION 3. *Let* $1 < c < d < \infty$. *Then there is an Orlicz function* $H(x) = H_{c,d}(x)$ *such that* $c \leqq xH'(x)/H(x) \leqq d$ *for* $0 < x \leqq 1$, *and for every Orlicz function* F *with* $c \leqq x F'(x)/F(x) \leqq d; 0 < x \leqq 1$, *there is a function equivalent to* F *in* E_H.

To prove the proposition we need first the following lemma.

LEMMA 2. *Let* $g(x)$ *be an Orlicz function on* $[0,1]$ *such that* $c \leqq xg'(x)/g(x) \leqq d$; $0 < x \leqq 1$, *and* $c > 1$. *Then there exists an Orlicz function* $G(x)$ *on* $[0,1]$ *such that*

1) G *is equivalent to* g
2) $G(1) = 1$; $G'(1) = c$
3) $c \leqq xG'(x)/G(x) \leqq d$; $0 < x \leqq 1$.

PROOF. We may clearly assume that $g(1) = 1$ and thus $x^c \leqq g(x) \leqq x^d$ for all $x \in [0,1]$. For $0 < x_1 < 1$ we define x_2 by the equation

$$c(g'(x_1)(x_2 - x_1) + g(x_1)) = g'(x_1)x_2.$$

It is easily checked that $x_1 \leqq x_2 \leqq x_1 c(d-1)/d(c-1)$; hence if we choose x_1 small enough we get that $x_2 < 1$. Define now $\tilde{g}(x)$ by

$$\tilde{g}(x) = \begin{cases} g(x) & 0 \leqq x \leqq x_1 \\ g(x_1) + g'(x_1)(x - x_1) \qquad ; & x_1 < x \leqq x_2 \\ [g(x_1) + g'(x_1)(x_2 - x_1)](x/x_2)^c & x_2 < x \leqq 1. \end{cases}$$

Then the function $G(x) = \tilde{g}(x)/\tilde{g}(1)$ has all the desired properties.

PROOF OF THE PROPOSITION. Let $\{F_n(x)\}_{n=1}^{\infty}$ be a dense sequence in the subset of $C(0,1)$ consisting of all Orlicz functions G which satisfy $G(1) = 1$, $G'(1) = c$ and $c \leqq xG'(x)/G(x) \leqq d$; $0 < x \leqq 1$. Let $t_n = 2^{-2^{n-1}}$; $n = 1,2,\cdots$ and define

$$H(x) = \begin{cases} F_1(x) & t_1 \leqq x \leqq 1 \\ H(t_n)F_n(x/t_n) \ ; & t_{n+1} \leqq x < t_n; \qquad n = 1,2,\cdots \\ 0 & x = 0 \end{cases}$$

Then, in view of Lemma 2, H has all the desired properties.

In terms of Orlicz sequence spaces, Proposition 3 shows the existence of universal elements.

COROLLARY. *For every* $1 < c < d \leqq \infty$ *there is an Orlicz function* $H(x)$ *with* $c \leqq xH'(x)/H(x) \leqq d;\ 0 < x \leqq 1$, *such that for any Orlicz function* $F(x)$ *with* $c \leqq xF'(x)/F(x) \leqq d;\ 0 < x \leqq 1$, l_F *is isomorphic to a complemented subspace of* l_H.

REMARKS. (1). It is clear that l_H is determined uniquely, up to isomorphism, by c and d. On the other hand H is not determined uniquely up to equivalence. Hence l_H does not have up to equivalence a unique symmetric basis.

(2) If $c^{-1} + d^{-1} = 1$, the space l_H obtained in the Corollary gives a non-trivial example of a space with a symmetric basis which is isomorphic to its conjugate.

4. Problems and comments

The results and examples obtained in [4] and in the previous sections lead naturally to some specific open problems as well as to some general directions in which further research seems to be desirable. This section is devoted to a discussion of such problems and research directions.

The most obvious question left open by our discussion is

PROBLEM 1. Assume l_f is isomorphic to a complemented subspace of l_F. Is f equivalent to a function in $E_{F,1}$?

A positive answer to Problem 1 would show that l_F is isomorphic to l_G iff $E_{G,1} = E_{F,1}$, up to equivalence and thus that l_F has up to equivalence a unique symmetric basis iff every G for which $E_{G,1} = E_{F,1}$ (up to equivalence) is already equivalent to F.

A Banach space X is called *prime* if every infinite dimensional complemented subspace of X is isomorphic to X. At present the only known prime spaces are c_0 and l_p; $1 \leqq p \leqq \infty$. The results and examples of Section 3 suggest that among the separable Orlicz sequence spaces there are new examples of prime spaces.

PROBLEM 2. Assume that F is a minimal Orlicz function. Is l_F a prime Banach space?

Problems 1 and 2 are special cases of the general question "what can be said about the structure of complemented subspaces of an Orlicz sequence space" Another aspect of this question which will probably play a role in the solution of Problem 2 is

PROBLEM 3. Let X be a complemented subspace of a separable Orlicz sequence space l_F. Does X have an unconditional basis?

Let us remark that in general such an X need not have a symmetric basis (e.g. X can be $l_p \oplus l_r$ with $p \neq r$).

In [4] we showed that for every Orlicz function F the set C_F contains x^p for some $p \geqq 1$. The set of numbers p such that $x^p \in C_F$ deserves further study. Let us note that if $l_f = l_p$ then Theorem 1 can be formulated in a simpler manner. The space l_p is isomorphic to a subspace of l_F iff $x^p \in C_F$. Indeed, assume that g is equivalent to x^p and $g \in C_{F,1}$. Then $C_g \subset C_F$ and by the result of [4] mentioned above $x^q \in C_g$ for some q. Clearly this q must be equal to p and hence $x^p \in C_F$.

Let us also note that the proof of Theorem 1 of [4] shows that if $f \in C_F$, then the following slightly stronger assertion than that of Theorem 1 here holds: For every $\varepsilon > 0$ there is a linear operator $T_\varepsilon : l_f \to l_F$ such that $(1-\varepsilon)\ \|x\| \leqq \|T_\varepsilon x\| \leqq (1+\varepsilon)\ \|x\|$, $x \in l_f$. (If such a situation holds it is said that l_f is almost isometric to a subspace of l_F). That this is the case is seen by taking the u, appearing in the proof of Theorem 1 of [4], to be sufficiently close to 1. The preceeding two observations show that if for some p and F, l_p is isomorphic to a subspace of l_F then l_p is almost isometric to a subspace of l_F.

We mention two specific problems (probably much easier than the preceeding ones) concerning the set of p's such that $x^p \in C_F$.*

PROBLEM 4. Let F be a reflexive Orlicz sequence space. Does there always exist a p such that l_p is isomorphic to a subspace as well as to a quotient space of l_F? (i.e. $x^p \in C_F$ and $x^q \in C_{F^*}$ where $q^{-1} + p^{-1} = 1$).

PROBLEM 5. Is the set of numbers p such that $x^p \in C_F$ always an interval?

Orlicz sequence spaces form a special subclass of the larger class of all Banach spaces with a symmetric basis. The statement of several of the results which we obtained for Orlicz sequence spaces make sense for this larger class. It is natural to

* See the note added in proof.

ask whether the results are still valid in the more general setting. Let us mention for example

PROBLEM 6. Does every Banach space X with a symmetric basis have a subspace isomorphic to c_0 or l_p for some $1 \leqq p < \infty$?

This question has been asked in several places for arbitrary Banach spaces. It seems very likely that for spaces with symmetric bases it will be easier to settle this problem.

There is a class of symmetric spaces that are not in general Orlicz spaces, which have received some attention in the literature (cf [1] and its references). Let $p \geqq 1$ and let $w = \{w_n\}$ be a decreasing sequence of positive numbers such that $\Sigma_n w_n = \infty$ and $\lim_n w_n = 0$. We denote by $\mu(w,p)$ the space of all sequences $\{x_n\} = x$ such that $\|x\|^p = \sup \Sigma_n |x_{\pi(n)}|^p w_n < \infty$ where the sup is taken over all permutations π (the sup is clearly attained by any permutation π for which $|x_{\pi(n)}| \geqq |x_{\pi(n+1)}|$; $n = 1,2,\cdots$). The space $\mu(w,p)$ is reflexive if $p > 1$. It is easy to verify that for $X = \mu(w,p)$ the answer to Problem 6 is positive in the following stronger form

PROPOSITION 4. *For every $p \geqq 1$ and every $w = \{w_n\}$ the space $\mu(w,p)$ has a complemented subspace isomorphic to l_p.*

PROOF. Choose a sequence of integers $0 = r_1 < r_2 \cdots$ such that $r_{n+1} > 2r_n + 1$ and

$$\sum_{i=r_n+1}^{r_{n+1}-r_n} w_i \geqq \sum_{i=1}^{r_n} w_i \ ; \qquad n = 1,2,\cdots$$

This is possible since $\Sigma_n w_n = +\infty$. Let e_i; $i = 1,2,\cdots$ denote the unit vectors of $\mu(w,p)$ and set

$$u_n = \sum_{i=r_n+1}^{r_{n+1}} e_i \Big/ \left(\sum_{i=1}^{r_{n+1}-r_n} w_i \right)^{1/p}; \qquad n = 1,2,\cdots$$

Obviously, the vectors u_n; $n = 1,2,\cdots$ form a normalized block basis of $\{e_i\}$ with constant coefficients.

Let λ_n; $n = 1,2,\cdots k$ be a finite sequence and let π be a permutation of the integers $\{1,2,\cdots,k\}$ such that

$$|\lambda_{\pi(n)}|^p / \sum_{i=1}^{r_{\pi(n)+1}-r_{\pi(n)}} w_i; \ n = 1,2,\cdots,k$$

forms a non-increasing sequence. Then

$$\Big\| \sum_{n=1}^{k} \lambda_n u_n \Big\|^p = \sum_{n=1}^{k} \left(|\lambda_{\pi(n)}|^p \Big/ \sum_{i=1}^{r_{\pi(n)+1}-r_{\pi(n)}} w_i \right) \cdot \sum_{i=1}^{r_{\pi(n)+1}-r_{\pi(n)}} w_{i+i_n}$$

where $i_n = \sum_{j=1}^{n-1} (r_{\pi(j)+1} - r_{\pi(j)})$. Since w_n is a nonincreasing sequence it follows immediately that

$$\Big\| \sum_{n=1}^{k} \lambda_n u_n \Big\| \leqq \left(\sum_{n=1}^{k} |\lambda_n|^p \right)^{1/p}.$$

On the other hand

$$\Big\| \sum_{n=1}^{k} \lambda_n u_n \Big\|^p \geqq \sum_{n=1}^{k} \left(|\lambda_n|^p \Big/ \sum_{i=1}^{r_{n+1}-r_n} w_i \right) \cdot \sum_{i=1}^{r_{n+1}-r_n} w_{i+r_n}$$

since $\sum_{j=1}^{n-1} (r_{j+1} - r_j) = r_n$. In view of our choice of r_n we have

$$\sum_{i=1}^{r_{n+1}-r_n} w_{i+r_n} \Big/ \sum_{i=1}^{r_{n+1}-r_n} w_i \geqq \sum_{i=r_n+1}^{r_{n+1}-r_n} w_i \Big/ \left(\sum_{i=1}^{r_n} w_i + \sum_{i=r_n+1}^{r_{n+1}-r_n} w_i \right) \geqq \tfrac{1}{2}; \quad n = 1,2,\cdots,k,$$

which implies

$$\Big\| \sum_{n=1}^{k} \lambda_n u_n \Big\| \geqq 2^{-1/p} \left(\sum_{n=1}^{k} |\lambda_n|^p \right)^{1/p}.$$

This shows that the closed subspace of $\mu(w,p)$ spanned by the vectors u_n; $n = 1,2,\cdots$ is isomorphic to l_p. This subspace is complemented in view of [5] Lemma 4 and therefore the proof is completed.

It is perhaps of interest to comment on the relation between the spaces $\mu(w,p)$ and the class of Orlicz sequence spaces. Since the only symmetric bases in an Orlicz sequence space are those which are induced by Orlicz functions, it follows that $\mu(w,p)$ is isomorphic to an Orlicz sequence space iff there is an Orlicz function F (satisfying, of course, the Δ_2 condition) such that for decreasing sequences of positive numbers $\{\lambda_n\}_{n=1}^{\infty}$, $\sum_n \lambda_n^p w_n < \infty$ iff $\sum_n F(\lambda_n) < \infty$. Sometimes, there exists such a function F. For example if $p \geqq 1$ and $w_n = (\log n)^{-1}$ then as is easily checked $F(x) = x^p / |\log x|$ has the desired property. On the other hand if $w_n = n^{-1}$ or more generally if e.g. $\liminf_{n\to\infty} S_{kn}/S_n = 1$ for every k where $S_n = \sum_{i=1}^{n} w_i$, then such an F does not exist. Indeed it follows easily that if a function F exists then the sequence $mF(S_m^{-1/p})$ $m = 1,2,\cdots$ is bounded and bounded away from 0. Using the Δ_2 condition for F, it is easy to deduce that for sufficiently large k $\liminf_{n\to\infty} S_{kn}/S_n$ is greater than 1.

It is clear that for no sequence $w = \{w_n\}$ (with $w_n \to 0$ as we always assume) is the function $F(x) = x^p$ a suitable function. Hence $\mu(w,p)$ is never isomorphic to l_p.

Note added in proof.

The answer to problems 4 and 5 is affirmative. Let $F(t)$ be an Orlicz function satisfying the Δ_2 condition and set

$$\alpha_F = \sup\{p; \sup_{0<x,t\leq 1} F(tx)/F(t)x^p < \infty\}$$

$$\beta_F = \inf\{p; \{p; \inf_{0<x\cdot t\leq 1} F(tx)/F(t)x^p > 0\}.$$

Then the following holds:

THEOREM *The space l_p is isomorphic to a subspace of l_F iff $p \in [\alpha_F, \beta_F]$.*

The interval $[\alpha_F, \beta_F]$ coincides with the interval associated to an Orlicz space l_F in several places in the literature (e.g. in [3]). The following corollary gives a strong answer to Problem 5.

COROLLARY. *Let l_F be a reflexive Orlicz sequence space. The space l_p is isomorphic to a subspace of l_F iff it is isomorphic to a quotient space of l_F.*

It can be shown that the interval associated to the space of Example 3 (in Section 3 above) is a single point.

Details will be published elsewhere.

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